

BOOK REMASTER · RESTORATION SAMPLE

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CHAPTER VII.

SUCCESSIVE DIFFERENTIATION.

Let us try the effect of repeating several times over the operation of differentiating a function (see p. 14). Begin with a concrete case.

Let $y = x^5$.

First differentiation,	$5x^4$.	
Second differentiation,	$5 \times 4x^3$	$= 20x^3$.
Third differentiation,	$5 \times 4 \times 3x^2$	$= 60x^2$.
Fourth differentiation,	$5 \times 4 \times 3 \times 2x$	$= 120x$.
Fifth differentiation,	$5 \times 4 \times 3 \times 2 \times 1$	$= 120$.
Sixth differentiation,		$= 0$.

There is a certain notation, with which we are already acquainted (see p. 15), used by some writers, that is very convenient. This is to employ the general symbol $f(x)$ for any function of x . Here the symbol $f()$ is read as “function of,” without saying what particular function is meant. So the statement $y = f(x)$ merely tells us that y is a function of x , it may be x^2 or ax^n , or $\cos x$ or any other complicated function of x .

The corresponding symbol for the differential coefficient is $f'(x)$, which is simpler to write than $\frac{dy}{dx}$. This is called the “derived function” of x .

Suppose we differentiate over again, we shall get the “second derived function” or second differential coefficient, which is denoted by $f''(x)$; and so on.

Now let us generalize.

Let $y = f(x) = x^n$.

First differentiation, $f'(x) = nx^{n-1}$.

Second differentiation, $f''(x) = n(n-1)x^{n-2}$.

Third differentiation, $f'''(x) = n(n-1)(n-2)x^{n-3}$.

Fourth differentiation,

$$f''''(x) = n(n-1)(n-2)(n-3)x^{n-4}.$$

etc., etc.

But this is not the only way of indicating successive differentiations. For,

if the original function be $y = f(x)$;

once differentiating gives $\frac{dy}{dx} = f'(x)$;

twice differentiating gives

$$\frac{d(\frac{dy}{dx})}{dx} = f''(x);$$

and this is more conveniently written as $\frac{d^2 y}{(dx)^2}$, or more usually $\frac{d^2 y}{dx^2}$. Similarly, we may write as

the result of thrice differentiating, $\frac{d^3 y}{dx^3} = f'''(x)$.

Examples.

Now let us try $y = f(x) = 7x^4 + 3 \cdot 5x^3 - \frac{1}{2}x^2 + x - 2$.

$$\frac{dy}{dx} = f'(x) = 28x^3 + 10 \cdot 5x^2 - x + 1,$$

$$\frac{d^2 y}{dx^2} = f''(x) = 84x^2 + 21x - 1,$$

$$\frac{d^3 y}{dx^3} = f'''(x) = 168x + 21,$$

$$\frac{d^4 y}{dx^4} = f''''(x) = 168,$$

$$\frac{d^5 y}{dx^5} = f'''''(x) = 0.$$

In a similar manner if $y = \phi(x) = 3x(x^2 - 4)$,

$$\phi'(x) = \frac{dy}{dx} = 3[x \times 2x + (x^2 - 4) \times 1] = 3(3x^2 - 4),$$

$$\phi''(x) = \frac{d^2 y}{dx^2} = 3 \times 6x = 18x,$$

$$\phi'''(x) = \frac{d^3 y}{dx^3} = 18,$$

$$\phi''''(x) = \frac{d^4 y}{dx^4} = 0.$$

Exercises IV. (See page 255 for Answers.)

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following expressions:

(1) $y = 17x + 12x^2$.

(2) $y = \frac{x^2 + a}{x + a}$.

(3)

$$y = 1 + \frac{x}{1} + \frac{x^2}{1 \times 2} + \frac{x^3}{1 \times 2 \times 3} + \frac{x^4}{1 \times 2 \times 3 \times 4}.$$

(4) Find the 2nd and 3rd derived functions in the Exercises III. (p. 46), No. 1 to No. 7, and in the Examples given (p. 41), No. 1 to No. 7.

CHAPTER XIX.

ON FINDING AREAS BY INTEGRATING.

One use of the integral calculus is to enable us to ascertain the values of areas bounded by curves.

Let us try to get at the subject bit by bit.

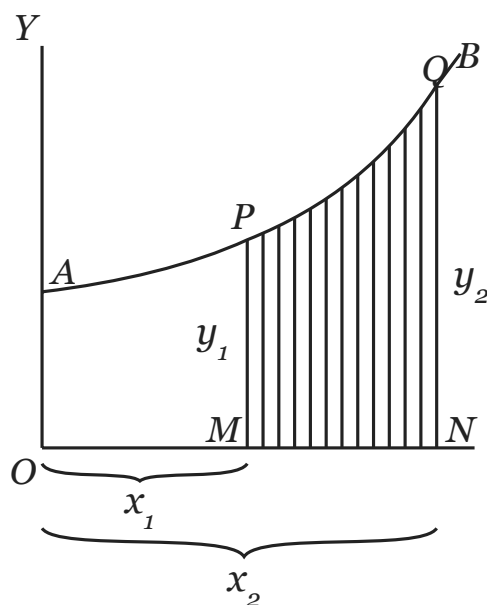


Fig. 52.

Let AB (Fig. 52) be a curve, the equation to which is known. That is, y in this curve is some known function of x . Think of a piece of the curve from the point P to the point Q .

Let a perpendicular PM be dropped from P , and another QN from the point Q . Then call $OM = x_1$ and $ON = x_2$, and the ordinates $PM = y_1$ and $QN = y_2$. We have thus marked out the area $PQNM$ that lies beneath the piece PQ . The problem is, *how can we calculate the value of this area?*

The secret of solving this problem is to conceive the area as being divided up into a lot of narrow strips, each of them being of the width dx . The smaller we take dx , the more of them there will be between x_1 and x_2 . Now, the whole area is clearly equal to the sum of the areas of all such strips.

Our business will then be to discover an expression for the area of any one narrow strip, and to integrate it so as to add together all the strips. Now think of any one of the strips. It will be like this:



being bounded between two vertical sides, with a flat bottom dx , and with a slightly curved sloping top. Suppose we take its *average* height as being y ; then, as its width is dx , its area will be $y \, dx$. And seeing that we may take the width as narrow as we please, if we only take it narrow enough its average height will be the same as the height at the middle of it. Now let us call the unknown value of the whole area S , meaning surface. The area of one strip will be simply a bit of the whole area, and may therefore be called dS . So we may write

$$\text{area of 1 strip} = dS = y \cdot dx.$$

If then we add up all the strips, we get

$$\text{total area } S = \int dS = \int y \, dx.$$

So then our finding S depends on whether we can integrate $y \cdot dx$ for the particular case, when we know what the value of y is as a function of x .

For instance, if you were told that for the particular curve in question $y = b + ax^2$, no doubt you could put that value into the expression and say: then I must find $\int (b + ax^2) \, dx$.

That is all very well; but a little thought will show you that something more must be done. Because the area we are trying to find is not the area under the *whole* length of the curve, but only the area limited on the left by PM , and on the right by QN , it follows that we must do something to define our area between those '*limits*.'

This introduces us to a new notion, namely that of *integrating between limits*. We suppose x to vary, and for the present purpose we do not require any value of x below x_1 (that is OM), nor any value of x above x_2 (that is ON). When an integral is to be thus defined between two limits, we call the lower of the two values the *inferior limit*, and the upper value the *superior limit*. Any integral so limited we designate as a *definite integral*, by way of distinguishing it from a *general integral* to which no limits are assigned.

In the symbols which give instructions to integrate, the limits are marked by putting them at the top and bottom respectively of the sign of integration. Thus the instruction

$$\int_{x=x_1}^{x=x_2} y \cdot dx$$